

Guo, James C.Y. (2006). "Kinematic Wave Unit Hydrograph for Storm Water Predictions", Vol 132, No. 4, ASCE J. of Irrigation and Drainage Engineering, July/August.

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## **INFLECTION POINT ON RECESSION HYDROGRAPH**

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Inflection point on the recession hydrograph has been recognized as one of the important time parameters when analyzing the watershed response to rainfall. Conventionally, the inflection point on a recession hydrograph is defined by the decay of flow rate (Ponce 1989). This paper presents a derivation to locate the inflection point by the normalized flow rate on the kinematic waver unit hydrograph (KWGH).

### **PARAMETERS AT INFLECTION POINT**

Overland flow produced by a sloping impervious plane under a uniform rainfall excess is kinematic in nature. The water flow depth can be described as:

$$\frac{dy}{dt} = i_e \quad (1)$$

where  $y$  =flow depth,  $i_e$ =rainfall excess, and  $t$ =time. At  $t=0$ , the sloping plane is assumed to be dry or  $y(t=0)=0$  everywhere. Under this dry bed condition, Eq 1 is integrated as:

$$y = i_e t \quad (2)$$

The kinematic wave associated with Eq 1 is propagated at the wave speed as:

$$\frac{dx}{dt} = V_w = \alpha \beta y^{\beta-1} = \alpha \beta i_e^{\beta-1} t^{\beta-1} \quad (3)$$

$$\alpha = k \sqrt{S_0} \quad (4)$$

where  $x$  = distance along waterway or sloping plane,  $V_w$  =kinematic wave speed,  $k$ = roughness of waterway,  $S_0$  = plane slope along waterway,  $\alpha$  = coefficient determined by plane slope and roughness, and  $\beta$ =exponent on rating curve. In practice,  $\beta=5/3$  for Manning's formula or  $3/2$  for Chezy's formula, Integrating Eq 3 yields

$$x = \frac{\alpha y^\beta}{i_e} \quad (5)$$

Eq's 2 and 5 defines the equilibrium water profile across the waterway at  $t=T_d$  in which  $T_d$  is rainfall duration. Eq 5 represents the basic rating curve relationship in kinematic wave.

$$Q_x = x i_e = \alpha y^\beta \quad (6)$$

Where  $Q_x$  = unit-width flow rate at Station  $x$ . After rain ceases,  $t > T_d$ , excess rainfall intensity equals to zero (i.e.  $i_e = 0$ ), therefore, Eq 1 becomes:

$$\frac{dy}{dt} = 0 \quad (7)$$

Integrating Eq 7 with respect to  $t$  gives  $y = \text{constant}$ . The receding wave is propagated with a constant depth by which the kinematic wave speed is constant as:

$$\frac{dx}{dt} = V_w = \alpha \beta y^{\beta-1} \quad (y = \text{constant after } T_d) \quad (8)$$

As illustrated in Figure 1, Eq 8 can be integrated for the receding wave to travel from Station  $x$  to Station  $X$  over the time interval from  $T_d$  to  $T$  as:

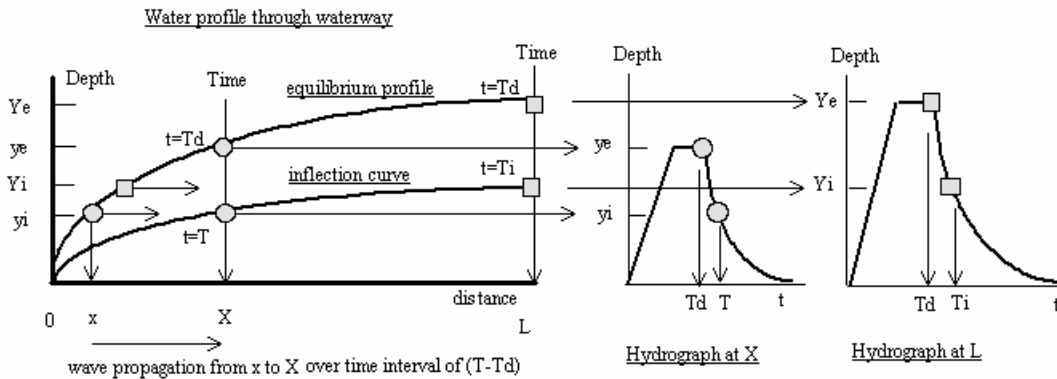
$$\int_x^X dx = \alpha \beta y \int_{T_d}^T dt \quad (9)$$

$$X = (\alpha \beta y^{\beta-1})(T - T_d) - x \quad (10)$$

Substituting Eq 4 into Eq 10 yields

$$X = (\alpha \beta y^{\beta-1})(T - T_d) - \frac{\alpha y^\beta}{i_e} \quad (11)$$

Eq 11 depicts the receding kinematic wave water profile. i.e. pair  $(X, y)$  at Station  $X$  at time  $T$ .



**Figure 1 Propagation of Receding Wave**

A recession hydrograph is defined by the variation of its curvature. The point of inflection is located where the curvature vanishes. Mathematically, such a location on the recession hydrograph can be determined by setting the second derivative of Eq 11 equal to zero.

$$\frac{d^2 X}{dy^2} = \left[ \alpha\beta(\beta-1)(\beta-2)y^{\beta-3}(T-T_d) \right] + \left[ \frac{\alpha\beta(\beta-1)}{i_e} y^{\beta-2} \right]_{y=Y_i, T=T_i} = 0 \quad (12)$$

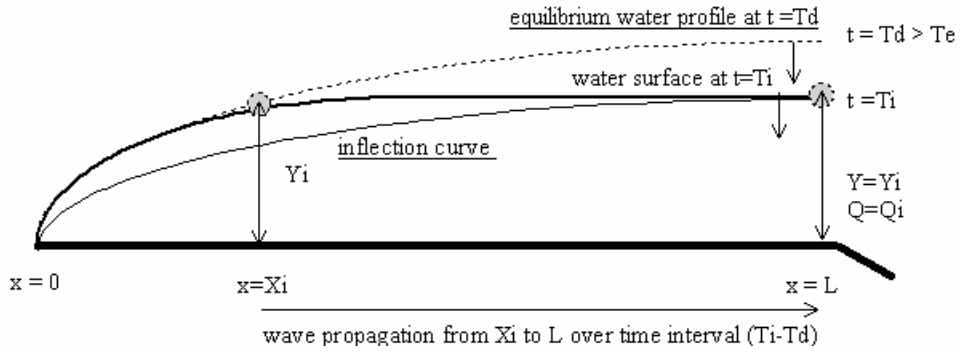
Rearranging Eq 12, the flow depth on the inflection point is obtained as:

$$Y_i = (T_i - T_d)(2 - \beta)i_e \quad (13)$$

Aided by Eq 11 with  $y=Y_i$ ,  $T=T_i$ , the corresponding location,  $x=X_i$ , is derived to be:

$$X_i = \left( \frac{\beta}{2 - \beta} + 1 \right) \frac{\alpha Y_i^\beta}{i_e} \quad (14)$$

Eq 14 defines the curve of inflection points along the waterway. As illustrated in Figure 2, the inflection point on the outlet hydrograph can be determined by Eq 14 with  $x=L$ .



**Figure 2 Illustration of Receding Water Profile**

Applying Eq 14 to the watershed outlet, the flow depth at the inflection point is determined as:

$$L = \left( \frac{\beta}{2 - \beta} + 1 \right) \left( \frac{\alpha Y_i^\beta}{i_e} \right) \quad (15)$$

where  $L$  = length of watershed or location of outlet and  $Y_i$  = flow depth at inflection point. According to the rating curve relationship used in the kinematic wave approach, we have

$$Q_i = \alpha Y_i^\beta \quad (16)$$

$$Q_E = i_e L \quad (17)$$

Where  $Q_i$  = unit-width flow rate at inflection point,  $Q_E$  = equilibrium or unit-width peak flow rate. Aided by Eq's 16 and 17, Eq 15 is rearranged as:

$$Q_i^* = \frac{Q_i}{Q_E} = 1 - \frac{\beta}{2} \quad (18)$$

where  $Q_i^*$  = dimensionless flow rate at inflection point. Next, normalize Eq 13 by the time of equilibrium, the time to inflection point is found to be:

$$T_{id}^* = \frac{T_i - T_d}{T_e} = \frac{Y_i}{(2 - \beta)i_e T_e} = \frac{Y_i}{(2 - \beta)Y_E} \quad (19)$$

where  $T_{id}^*$  = dimensionless time to inflection point after rain ceases,  $T_i$  = time to the inflection point after rain ceases,  $T_d$  = rainfall duration, and  $T_e$  = time of equilibrium for waterway. Aided by the rating curve in Eq 16 and the flow ratio in Eq 18, Eq 19 is converted into

$$T_{id}^* = \frac{1}{(2 - \beta)} \left( \frac{Q_i}{Q_E} \right)^{\frac{1}{\beta}} = \frac{(1 - \frac{\beta}{2})^{\frac{1}{\beta}}}{2 - \beta} \quad \text{after rain ceases} \quad (20)$$

Using the beginning of the rainfall event as the base, the time to inflection point is calculated as:

$$T_i^* = T_{id}^* + \frac{T_d}{T_e} \quad \text{after rain starts} \quad (21)$$

where  $T_{io}^*$  = dimensionless time to inflection point after rain starts. Substituting Eq 7 into Eq 6 yields

$$Y_i^* = \frac{Y_i}{Y_E} = (1 - \frac{\beta}{2})^{\frac{1}{\beta}} \quad (22)$$

where  $Y_i^*$  = dimensionless flow depth at inflection point. As discussed in the original paper, the rainfall duration for the KWUH is set to be the time of equilibrium. With  $T_d = T_e$ , the values of these dimensionless variables at the inflection point is summarized in Table 1 using different formulas:

Formula	$\beta$	$T_{id}^*$	$T_i^*$	$Q_i^*$	$Y_i^*$
Manning's	5/3	1.024	2.024	0.167	0.341
Chezy's	3/2	0.658	1.658	0.333	0.439

**Table 1 Values of Dimensionless Variables at Inflection Point**

## CLOSURE

1. As indicated in Table 1, the recession limb of the KWUH decays faster using Manning's formula than that using Chezy's formula. Conversely, Manning's formula gives a longer recession hydrograph.

2. Guo (2006) reported that the time to inflection point on the recession limb of the KWUH was graphically determined to be 2.28  $T_e$  measured from the beginning of the event. As shown in Table 1, the time to inflection point is 2.024 or 1.658  $T_e$ , depending on the empirical formula.

3. Under a uniform rainfall excess, the time of concentration is numerically equal to the time of equilibrium. Without rigorous mathematical proof, it has been suggested that the time of concentration be the time between the center of the mass of rainfall excess and the inflection point on the recession limb of the runoff hydrograph (McCuen 1998, Guo 2001). This paper has demonstrated that the value of  $T_i^*$  is 1.024 or close to unit.

$$T_i^* = \frac{T_i - T_d}{T_e} = 1.024 \quad (23)$$

$$T_i - T_d = 1.024T_e \approx T_e \quad (24)$$

As indicated in Eq 24, the time of concentration or equilibrium is closely equal to the time to inflection point measured after the rain ceases, instead of the center of the rainfall excess.

## **APPENDIX I**

Guo, James C.Y. (2006). "Kinematic Wave Unit Hydrograph for Storm Water Predictions", Vol 132, No. 4, ASCE J. of Irrigation and Drainage Engineering, July/August.

Guo, James C.Y. (2001). "Rational Hydrograph Method for Small Urban Catchments," ASCE J. of Hydrologic Engineering, Vol 6, No.4, July/August.

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## **APPENDIX II**

$i_e$  = rainfall excess

L = length of watershed or location of outlet,

$Q_i$  = flow rate at inflection point,

$Q_E$  = equilibrium or peak flow rate when rain ceases

$T_d$  = rainfall duration,

$T_e$  = time of equilibrium for the waterway

$T_i$  = time to inflection point after rain starts

$T_{id}^*$  = dimensionless time to inflection point after rain ceases.

$Y_i$  = flow depth at inflection point,

$\alpha$  = coefficient determined by slope and roughness of the plane,

$\beta=5/3$  for Manning's formula or  $3/2$  for Chezy's formula,

\*= dimensionless variable